SET CONCEPTS I

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Abstract. This is a introduction to the initial concepts of set, elements, functions, and relations. The ideas contained herein can be put on a solid foundation using an axiomatic approach to set theory. For the time being, we eschew this technical development in the interest of more rapidly attaining the important ideas behind functions and relations.

Our goal is to build enough tools to briefly define the set natural numbers, to thoroughly develop the integers from the natural numbers, to define modulo arithmetic, and to construct the rational numbers from the integers.

1. Sets and Elements

Intuitively, a *set* is a collection of *elements*. We should not think of a set as a "container", but rather as the elements themselves. We assume that we can distinguish between different elements, and that we can determine whether or not a given element is in a given set.

The relationship of two elements a and b being the same is *equality* and is denoted $a = b$. The negation of this relation is denoted $a \neq b$, that is, $a \neq b \Leftrightarrow \neg(a = b)$.

The relationship of an element a being a member of a set A *containment* and is denoted $a \in A$. The negation of this relation is denoted $b \notin A$, that is, $b \notin A \Leftrightarrow \neg(b \in A)$.

A set is determined by the elements it contains. That is, two sets are considered equal if and only if they contain the same elements:

$$A = B \Leftrightarrow (a \in A \Leftrightarrow a \in B).$$

One way of describing a set is by explicitly listing its members. Such lists are surrounded by braces, e.g., the set of the first five prime integers is $\{2, 3, 5, 7, 11\}$. If the pattern is clear, we may use dots; for example, to label the set of all prime numbers as P, we may write $P = \{2, 3, 5, 7, 11, 13, \dots\}$. Thus $2 \in P$ and $23 \in P$, but $1 \notin P$ and $21 \notin P$. As another example, if we denote the set of all integers by $\mathbb{Z}$, we may write $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Note that the order of elements in a list is irrelevant in determining a set, for example, $\{5, 3, 7, 11, 2\} = \{2, 3, 5, 7, 11\}$. Also, there is no such thing as the "multiplicity" of an element in a set, for example $\{1, 3, 2, 2, 1\} = \{1, 2, 3\}$.

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2. Subsets

If A and B are sets and all of the elements in A are also contained in B, we say that A is a *subset* of B or that A is *included* in B and write $A \subset B$:

$$A \subset B \Leftrightarrow (a \in A \Rightarrow a \in B).$$

For example, $\{1, 3, 5\} \subset \{1, 2, 3, 4, 5\}$. Note that any set is a subset of itself. We say that A is a *proper subset* of B is $A \subset B$ but $A \neq B$.

It follows immediately from the definition of subset that

$$A = B \Leftrightarrow (A \subset B \wedge B \subset A).$$

Thus to show that two sets are equal, it suffices to show that each is contained in the other.

A set containing no elements is called the *empty set* and is denoted $\varnothing$. Since a set is determined by its elements, there is only one empty set. Note that the empty set is a subset of any set.

3. Set Operations

We may construct new sets as subsets of existing sets by specifying properties. Specifically, we may have a proposition $p(x)$ which is true for some elements x in a set X and not true for others. Then we may construct the set

$$\{x \in X \mid p(x) \text{ is true}\};$$

this is read "the set of x in X such that $p(x)$". The construction of this set is called *specification*. For example, if we let $\mathbb{Z}$ be the set of integers, the set P of all prime numbers could be specified as $P = \{n \in \mathbb{Z} \mid n \text{ is prime}\}$.

Let A and B be subsets of some "universal set" U and define the following set operations:

$$\begin{aligned} \text{Intersection:} \quad & A \cap B = \{x \in U \mid x \in A \wedge x \in B\} \\ \text{Union:} \quad & A \cup B = \{x \in U \mid x \in A \vee x \in B\} \\ \text{Complement:} \quad & A \smallsetminus B = \{x \in U \mid x \in A \wedge x \notin B\} \end{aligned}$$

The pictures which correspond to these operations are called *Venn diagrams*.

Example 1. Let $A = \{1,3,5,7,9\}$, $B = \{1,2,3,4,5\}$. Then $A \cap B = \{1,3,5\}$, $A \cup B = \{1,2,3,4,5,7,9\}$, $A \smallsetminus B = \{7,9\}$, and $B \smallsetminus A = \{2,4\}$. $\square$

Example 2. Let A and B be two distinct nonparallel lines in a plane. We may consider A and B as a set of points. Their intersection is a single point, their union is crossing lines, and the complement of A with respect to B is A minus the point of intersection. $\square$

If $A \cap B = \varnothing$, we say that A and B are *disjoint*.

Example 3. A *sphere* is the set of points in space equidistant from a given point, called its *center*; the common distance to the center is called that *radius* of the sphere. Thus a sphere is the surface of a solid ball.

Take two points in space such that the distance between them is 10, and imagine two spheres centered at these points. Let one of the spheres have radius 5. If the radius of the other sphere is less than 5 or greater than 15, then the spheres are disjoint. If the radius of the other sphere is exactly 5 or 15, the intersection is a single point. If the radius of the other sphere is between 5 and 15, the spheres intersect in a circle. $\square$

The following properties are sometimes useful in proofs:

- $A = A \cup A = A \cap A$
- $\varnothing \cap A = \varnothing$
- $\varnothing \cup A = A$
- $A \subset B \Leftrightarrow A \cap B = A$
- $A \subset B \Leftrightarrow A \cup B = B$

As an example, we prove one of these properties.

Proposition 1. *Let A and B be a sets. Then $A \subset B \Leftrightarrow A \cap B = A$.*

Proof. To prove an if and only if statement, we prove implication in both directions.

($\Rightarrow$) Assume that $A \subset B$. We wish to show that $A \cap B = A$. To show that two sets are equal, we show that each is contained in the other.

($\subset$) To show that $A \cap B \subset A$, it suffices to show that every element of $A \cap B$ is in A. Thus we select an arbitrary element $c \in A \cap B$ and show that it is in A. Now by definition of intersection, $c \in A \cap B$ means that $c \in A$ and $c \in B$. Thus $c \in A$. Since c was arbitrary, every element of $A \cap B$ is contained in A. Thus $A \cap B \subset A$.

($\supset$) Let $a \in A$. We wish to show that $a \in A \cap B$. Since $A \subset B$, then every element of A is an element of B. Thus $a \in B$. So $a \in A$ and $a \in B$. By definition of intersection, $a \in A \cap B$. Thus $A \subset A \cap B$.

Since $A \cap B \subset A$ and $A \subset A \cap B$, we have $A \cap B = A$.

($\Leftarrow$) Assume that $A \cap B = A$. We wish to show that $A \subset B$. Let $a \in A$. It suffices to show that $a \in B$. Since $A \cap B = A$, then $a \in A \cap B$. Thus $a \in A$ and $a \in B$. In particular, $a \in B$. $\square$

Now let us prove the analogous statement in compressed form.

Proposition 2. *Let A and B be a sets. Then $A \subset B \Leftrightarrow A \cup B = B$.*

Proof.
($\Rightarrow$) Assume that $A \subset B$. Clearly $B \subset A \cup B$, so we show that $A \cup B \subset B$. Let $c \in A \cup B$. Then $c \in A$ or $c \in B$. If $c \in B$ we are done, so assume that $c \in A$. Since $A \subset B$, then $c \in B$ by definition of subset. Thus $A \cup B \subset B$.

($\Leftarrow$) Assume that $A \cup B = B$ and let $a \in A$. Thus $a \in A \cup B$, so $a \in B$. Thus $A \subset B$. $\square$

The following properties state that union and intersection are commutative and associative operations, and that they distribute over each other. These properties are intuitively clear via Venn diagrams, and can be proved rigorously from the definitions of intersection and union with the help of truth tables.

- $A \cap B = B \cap A$
- $A \cup B = B \cup A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Since $(A \cap B) \cap C = A \cap (B \cap C)$, parentheses are useless and we write $A \cap B \cap C$. This extends to four sets, five sets, and so on. Similar remarks apply to unions.

The following properties of complement are known as *DeMorgan's Laws*. You should draw Venn diagrams of these situations to convince yourself that these properties are true (however, these diagrams should not be considered as proofs).

- $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$
- $A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$

Here are a few more properties of complement:

- $A \subset B \Rightarrow A \cup (B \smallsetminus A) = B$;
- $A \subset B \Rightarrow A \cap (B \smallsetminus A) = \varnothing$;
- $A \smallsetminus (B \smallsetminus C) = (A \smallsetminus B) \cup (A \cap B \cap C)$;
- $(A \smallsetminus B) \smallsetminus C = A \smallsetminus (B \cup C)$.

4. Cartesian Product

Let a and b be elements. The *ordered pair* of a and b is denoted (a, b) and is defined as

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

This is the technical definition; think about how it relates to the intuitive approach below.

Intuitively, if a and b are elements, the *ordered pair* with first coordinate a and second coordinate b is something like a set containing a and b, but in such a way that the order matters. We denote this ordered pair by (a, b) and declare that it has the following "defining property":

$$(a, b) = (c, d) \Leftrightarrow (a = c) \wedge (b = d).$$

The *cartesian product* of the sets A and B is denoted $A \times B$ and is defined to be the set of all ordered pairs whose first coordinate is in A and whose second coordinate is in B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Example 4. Let $A = \{1, 3, 5\}$ and let $B = \{1, 4\}$. Then

$$A \times B = \{(1, 1), (1, 4), (3, 1), (3, 4), (5, 1), (5, 4)\}.$$

In particular, this set contains 6 elements. $\square$

In general, if A contains m elements and B contains n elements, where m and n are natural numbers, then $A \times B$ contains mn elements.

Similarly, we have *ordered triples* (a, b, c), with a "defining property"

$$(a, b, c) = (d, e, f) \Leftrightarrow (a = d) \wedge (b = e) \wedge (c = f).$$

The we declare the cartesian product of three sets to be

$$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}.$$

By slight of hand which we will not discuss at this point, one may show that it is possible to "identify" the order pair $((a, b), c)$ with the ordered pair $(a, (b, c))$, so that $(A \times B) \times C$ is identified with $A \times (B \times C)$, and that both of these are "identified" with $A \times B \times C$. This forces a kind of associativity on the operation of cartesian product.

We continue with *ordered n-tuples* and the cartesian product of n sets, for any natural number n. If A is a set, the cartesian product of A with itself n times is denoted A^n. For example, $A^2 = A \times A$ and $A^3 = A \times A \times A$. The entries of an ordered n-tuple in such a cartesian product are called *coordinates*.

We have the following properties:

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$;
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

As an example, we prove one of these properties.

Proposition 3. *Let A, B, C, and D be sets.*
Then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. We use the defining property of an ordered pair to show equality of sets by showing containment in both directions.

($\subset$) Let $\alpha \in (A \times B) \cap (C \times D)$. Then $\alpha \in A \times B$ and $\alpha \in C \times D$. Then $\alpha = (a, b)$, where $a \in A$ and $b \in B$, and $\alpha = (c, d)$, where $c \in C$ and $d \in D$. Since $(a, b) = (c, d)$, we have $a = c$ and $b = d$.

Now $a \in A$ and $a = c \in C$, so $a \in A \cap C$. Also $b \in B$ and $b = d \in D$, so $b \in B \cap D$. Therefore $(a, b) \in (A \cap C) \times (B \cap D)$.

($\supset$) Let $\alpha \in (A \cap C) \times (B \cap D)$. Then $\alpha = (x, y)$, where $x \in A \cap C$ and $y \in B \cap D$. Thus $x \in A$ and $x \in C$. Also $y \in B$ and $y \in D$. So $(x, y) \in A \times B$ and $(x, y) \in C \times D$. Therefore $(x, y) \in (A \times B) \cap (C \times D)$. □

5. Numbers

Later, we will formally develop some of the standard number systems. For the time being, we use these familiar sets only in examples. Since they are useful for intuition into general set constructions, at this time we specify the standard names for the common sets of numbers.

The following sets of numbers are standard:

Natural Numbers:	$\mathbb{N} = \{0, 1, 2, 3, \dots\}$
Integers:	$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
Rational Numbers:	$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$
Real Numbers:	$\mathbb{R} = \{\text{Gaps in } \mathbb{Q}\}$
Complex Numbers:	$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$

We view $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

The following standard notation gives subsets of the real numbers, called *intervals*:

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ (closed)
- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ (open)
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$ (closed)
- $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$ (open)
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$ (closed)
- $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$ (open)

Example 5. Let $A = [1, 5]$ be the closed interval of real numbers between 1 and 5 and let $B = (10, 16)$ be the open interval of real numbers between 10 and 16. Let $C = A \cup B$. Let $\mathbb{N}$ be the set of natural numbers. How many elements are in $C \cap \mathbb{N}$?

Solution. The set $C \cap \mathbb{N}$ is the set of natural numbers between 1 and 5 inclusive and between 10 and 16 exclusive. Thus $C \cap \mathbb{N} = \{1, 2, 3, 4, 5, 11, 12, 13, 14, 15\}$. Therefore $C \cap \mathbb{N}$ has 10 elements. $\square$

The first three of our standard sets of numbers, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, have an algebraic nature; they are the minimum sets of numbers which allow us to add and multiply ($\mathbb{N}$), subtract ($\mathbb{Z}$), and divide ($\mathbb{Q}$).

The real numbers are the *geometric completion* of the rational numbers, constructed from the rational numbers by filling in the gaps. For example, the sequence

$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1414213, \dots\}$$

consists of rational numbers but converges to $\sqrt{2}$, which is not a rational number. The rational number line has "holes" where the irrational numbers belong, and for this reason it does not model the synthetic notion of a line as well as the real numbers.

We think of a point as zero-dimensional space. A set which represents zero-dimensional space is $\{0\}$. A line is one-dimensional space, and is represented by $\mathbb{R}$. A plane is two-dimensional space, and is represented by $\mathbb{R}^2$, the set of all ordered pairs of real numbers. Three-dimensional space is represented by $\mathbb{R}^3$, the set of all ordered triples of real numbers.

The complex numbers are the *algebraic closure* of the real numbers, and were developed from the real numbers so that all polynomials may be factored.

Example 6. Let $A = [1, 3]$, $B = [3, 8]$, and $C = (0, 3)$ be intervals of real numbers. The set $A \times B \times C$ forms a cube in $\mathbb{R}^3$, which is closed on its sides (it contains its boundary there) but open on the top and bottom (it does not contain its boundary there). How many elements are in $(A \times B \times C) \cap (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$?

Solution. By generalizing a previous proposition, we have

$$(A \times B \times C) \cap (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) = (A \cap \mathbb{Z}) \times (B \cap \mathbb{Z}) \times (C \cap \mathbb{Z}).$$

Now $A \times \mathbb{Z} = \{1, 2, 3\}$, $B \times \mathbb{Z} = \{3, 4, 5, 6, 7, 8\}$, and $C \times \mathbb{Z} = \{1, 2\}$. Thus $(A \times B \times C) \cap (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$ has $3 \cdot 6 \cdot 2 = 36$ elements. $\square$

Warning 1. The notation for ordered pair (a, b) is the same as the standard notation for open interval of real numbers, but its meaning is entirely different. This is standard, and you must decide from the context which meaning is intended.

6. Exercises I

Exercise 1. Let A, B, and C be the following subsets of $\mathbb{N}$:

- $A = \{n \in \mathbb{N} \mid n < 25\}$;
- $E = \{n \in A \mid n \text{ is even}\}$;
- $O = \{n \in A \mid n \text{ is odd}\}$;
- $P = \{n \in A \mid n \text{ is prime}\}$;
- $S = \{n \in A \mid n \text{ is a square}\}$;

Compute the following sets:
(a) $(E \cap P) \cup S$;
(b) $(E \cap S) \cup (P \setminus O)$;
(c) $P \times S$;
(d) $(O \cap S) \times (E \cap S)$.

Exercise 2. In each case, draw a Venn diagram representing the situation:
(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$;
(b) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;
(c) $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

Exercise 3. Let A and B be subsets of a set U. The *symmetric difference* of A and B, denoted $A \triangle B$, is the set of points in U which are in either A or B but not in both.
(a) Draw a Venn diagram describing $A \triangle B$.
(b) Find two set expressions which could be used to define $A \triangle B$, and justify your answer.
(c) Choose one of your expressions above as a formal definition, and use it to prove that symmetric difference is commutative and associative. Your proof here may use the fact that intersection and union are commutative and associative without proving these facts.

In the next two exercises, you should read "show that" to mean "give a formal proof that".

Exercise 4. Let A, B, and C sets. Show that

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Exercise 5. Let A, B, and C be sets. Show that

$$(A \cup B) \times C = (A \times C) \cup (B \times C).$$

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